A NOTE ON MONOTONE BASES

BY

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ABSTRACT

In a finite-dimensional Banach space with a monotone basis, the range of a norm-1 projection has a monotone basis.

Pełczyński [2] proved that for any finite-dimensional Banach space E and any $\lambda > 1$ there is a finite-dimensional Banach space F with a basis of constant $\leqq \lambda$ and a projection Q of norm 1 in F such that E is isometric to QF. He also showed that we can get $\lambda = 1$ if F is not required to be finite-dimensional, and asked whether $\lambda = 1$ can always be attained with finite-dimensional F. The following result answers this question in the negative.

THEOREM. *Let F be a finite-dimensional Banach space with a monotone basis, and let Q be a norm-1 projection in F. Then $E \equiv QF$ has a monotone basis.*

For proof we shall need the following version of the ergodic theorem (cf. [1], VIII 5.2–3);

LEMMA. *Let X be a Banach space, and T a finite rank operator in X with $\| T \| \leqq 1$. Then there is a norm-1 projection of X onto the finite-dimensional subspace $\{x \in X;\ Tx = x\}$.*

PROOF OF THE THEOREM. By assumption, $I_F = \Sigma_{i=1} A_i$ where the A_i are rank 1 projections, and $\| \Sigma_{i=1}^{k} A_i \| = 1, (k = 1, 2, \cdots, n)$. Let m be the last index for which $R \equiv QA_{m|E} \neq 0$, and let $T = I_E - R$. Then $\| T \| \leqq \| Q . \Sigma_{i=1}^{m-1} A_i \| \leqq 1$. Since R is a rank 1 operator, $E_1 \equiv \{x \in E;\ Tx = x\} = \ker(R)$ is of codimension 1 in E. By the Lemma, there is a norm-1 projection P_1 of E onto E_1. Now, $P_1 Q$ is a norm-1 projection of F onto E_1, and so repeating the procedure $d = \dim(E)$

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times, we get norm-1 projections onto E_2, $E_3, \cdots\cdots, E_d = 0$, where $E \supset E_1 \supset, \cdots$, $E_2 \supset \cdots \supset E_d$ have dimensions decreasing by 1. This shows that E has a monotone basis (cf e.g. [3], p. 249).

REMARK. The existence of finite-dimensional Banach spaces without monotone bases is well known (cf. [3] Ch.I, 1).

REFERENCES

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2. A. Pełczyński, *Any separable Banach space with the bounded approximation property is a complemented subspace pf a Banach space with a basis*, Studia Math. **40** (1971), 239–243.

3. I. Singer, *Bases in Banach spaces*, I, Springer, 1970.

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